CHAPTER

SIX

PERCOLATION AND RANDOM BOUNDARIES

6.1. CLUSTERS AND PERCOLATION

In Chapter 5 we have mentioned a model of a ball lightning in which the active matter distribution forms a fractal cluster. The fractal properties of the matter distribution make the combination of a small weight (almost the whole volume of the ball lightning is free of active matter) and great strength (even though the active matter occupies a small fraction of volume, it penetrates to all parts of the lightning volume) possible. It is clear that a suitable fractal dimension of the cluster is not sufficient to reach the required strength of the construction. For a given fractal dimension, the matter would be concentrated in numerous relatively small and weakly connected clumps. The strength of the whole construction would be insufficient. Apart from the fractal dimension, the degree of internal connection plays an important role. It would be too restrictive to require that all particles of the active matter are in contact with one another (this is how the connection is understood in topology): it is admissible that there are a few pieces or clumps within a ball lightning. A more realistic picture is one that requires that there are connected parts whose sizes are of the order of the overall size of the ball lightning. Now we can distract ourselves from the fact that a ball lightning has a finite size and we consider the problem at scales much smaller than the lightning size. This brings us to an idealized description of the fractal cluster of active matter within a ball lightning: to ensure its strength, connected components that stretch from infinity to infinity must be present. Thus, we have approached an important concept of modern mathematics — percolation.

In our discussion, percolation appears to be connected to the model of a ball lightning. Of course, this is not the most practical application of the percolation theory. More important are the applications to the theory of compositional materials which consist of a mixture of ingredients with various properties. Consider the conductivity of a sample made of a pressed mixture of pieces of conductors and insulators. Clearly, the conductivity essentially depends very much on whether or not the electric current can flow along the system of conducting pieces. This phenomenon gives rise to the term "percolation" applied in various areas of science. Another important application of the percolation theory is in cosmology where the percolation properties of matter distribution in the universe are analyzed.

Mentioning the example of a composite material pressed from a metal and a dielectric, we imply that we have a sufficiently big, homogeneous piece in which the contents are oriented, on the average, isotropically and placed randomly. In a general case, the percolation theory usually considers the percolation properties of random sets in an infinite space when probability distributions, which determine these sets, are homogeneous and isotropic.

Nowadays, acceptable conceptions of the properties of percolation objects have been formed. In a popularized form, they are summarized by Efros (1982). First, it seems that the possibility of percolation depends only weakly on the shape and detailed structure of elementary objects forming a percolating cluster (pieces of metal in the example above). This hypothesis, known as the hypothesis of universality, is plausible as long as the size of the percolating cluster is much larger than the size of the individual pieces of metal. Next, it is clear that the percolation properties essentially depend on whether the considered object is one-dimensional or three-dimensional. In one-dimensional objects, percolation is practically impossible: any small particle of an ideal insulator disconnects the one-dimensional electrical circuit. In two dimensions, percolation over the metal phase implies impossibility of percolation over the insulator phase and vice versa, provided that the considered system is statistically isotropic. It is then natural to assume that in this case percolation is possible over the phase which occupies more than half of the area. In three dimensions, percolation can occur over both phases simultaneously. One can therefore expect that for small concentrations of the metal phase percolation over this phase is impossible, for higher concentration percolation occurs over both phases, and for still higher concentration of the metal phase percolation occurs over this phase exclusively. When the statistical properties of both the metal and dielectric phases depend identically on concentration, the thresholds for percolation (i.e., the value of concentration C_m for which percolation begins over the metal phase and the value C_d for which percolation ceases over the dielectric phase) are connected by an obvious relation

$$C_m = 1 - C_d .$$

Numerical simulation gives $C_m \approx 0.16$. This important figure will be frequently used below. Thus, as little as about twenty percent of the conductivity admixture make a composite material conductive. When the fraction of metal reaches approximately eighty percent, percolation over the dielectric becomes impossible; it becomes isolated into separated pieces.

The theory of percolation is now a vast field of mathematics and physics where the results are based on rigorous theorems as well as on numerical simulations and physical experiments. During the last thirty years, the pioneering work of English physicist and engineer Brodbent and mathematician Hammersley has given birth to an immense number of papers. In order to give a more concrete idea of applications of these results, we consider an example of the percolation properties of magnetic field lines (Zeldovich, 1983).

Importance of magnetic lines, which are determined by the equation

$$d\mathbf{r} = \mathbf{B}(\mathbf{r})\,d\alpha \tag{1}$$

(where α is an arbitrary scalar parameter), has been revived in recent decades, many years after their introduction by M. Faraday, mainly in connection with the problem of controlled thermonuclear fusion and problems of astrophysics, where the magnetic field is frozen into a plasma. In this case the plasma distinguishes a natural reference frame in three-dimensional space, comoving with the fluid, in which a (pseudo) vector $\mathbf{B}$ is determined (in contrast to the electromagnetic field tensor F_{ik} in empty Minkowski's space where there is no such frame).

Let us consider the topological properties of systems of magnetic lines. One of the important questions is: can magnetic lines extend to infinity in this or that direction? One can replace magnetic lines by narrow pipes filled with a liquid and ask whether or not a given system of pipes can transport the liquid to infinite distances. This percolational statement of the problem is complementary to the analysis of linkages of magnetic lines thoroughly developed by Moffatt and other authors (see Moffatt, 1969; Zeldovich *et al.*, 1983; Ruzmaikin and Sokoloff, 1980).

Percolation along magnetic lines is interesting mainly because the charged particles are spiralling along these lines. Evidently, the thermal conductivity of fusion plasmas, as determined by diffusion of electrons, depends on the properties of the magnetic lines and the surfaces around which the lines are wound. In particular, Kadomtsev and Pogutse (1979) consider a three-dimensional problem in which a weak random two-dimensional magnetic field $\mathbf{b} = (b_x, b_y, 0)$ is superimposed on a strong uniform field $\mathbf{B} = (0, 0, B_0)$ directed along the z-axis. Diffusion and heat conduction are determined by the tangling of magnetic lines associated with the presence of b. The two-dimensional field b can be expressed through a scalar function a of two variables, i.e., the z-component of the vector potential $\mathbf{a} = (0, 0, a)$:

$$b_x = \frac{\partial a}{\partial y}; \quad b_y = \frac{-\partial a}{\partial x}. \tag{2}$$

On the plane (x, y), the magnetic lines wound around the maxima in a counter-clockwise direction and around the minima in a clockwise direction.

When a is independent of both the time t and the z-coordinate, the problem reduces to the percolation properties of a random function of two variables.

Let us adopt the following normalization:

$$\langle a \rangle = 0; \quad \langle a^2 \rangle = 1. \tag{3}$$

Let us also consider the sufficiently smooth functions a, whose autocorrelation function is also sufficiently small at large distances. Using a spectral representation, this means that the Fourier transform of a has random phases and amplitudes a_k such that, e.g.,

$$\langle a_k^2 \rangle \sim \exp\left(-\frac{k^2}{k_0^2}\right), \quad k > k_0\,,$$

and (4)

$$\langle a_k^2 \rangle \sim \left(\frac{k}{k_0}\right)^{2n}, \quad n > 0, \quad k \ll k_0\,.$$

It is natural to expect that the regions where $a > \varepsilon$ (with $\varepsilon > 0$), which occupy less than half the total area, are isolated (islands) and there is no percolation over them. One may also introduce the quantity $l(\varepsilon)$ that characterizes the average size of an island or the average length of the isoline that surrounds an island.

In a two-dimensional problem, when the plane is divided into regions of two types, it is natural to assume that when the regions of one type form isolated islands, the regions of the other type form a globally connected ocean. Correspondingly, the conditions $a < \varepsilon$ and $\varepsilon > 0$ determine a unified region along which percolation occurs.[a] The value $\varepsilon = 0$ is critical with respect to the possibility of percolation.

Kadomtsev and Pogutse (1979) obtained an estimate of the function $l(\varepsilon)$ and suggested that the properly averaged value of $l(\varepsilon)$ plays the role of the effective free pathlength in the theory of diffusion and heat conduction.

In plasma devices with strong longitudinal field $B_z = B_0$, there is no reason to believe that the weak perturbations, **b**, are independent of z and/or time. Let us consider an opposite case when (i) either the field B_0 directed along z is absent or there is periodicity along the z-axis, with period $2\pi R$ that corresponds to a torus of large radius R with the z-axis chosen along the torus' large circumference; (ii) the two-dimensional field **b** is independent of both z and t; and (iii) the total field is strong and

[a] Some parts of the region where $a < \varepsilon$ can form lakes isolated from the percolating ocean.

charged particles move along the magnetic lines, i.e., the Larmor radius r of the electrons' spiral trajectories is neglected as well as the collisions and other effects that can cause jumps of the electrons from one magnetic line to another.

Due to this change of the magnetic field configuration, the solution of the considered problem becomes quite different from that obtained by Kadomtsev and Pogutse (1979): in a two-dimensional random field, diffusion of electrons is impossible, i.e., diffusivity is zero. To express this result in a constructive form, this means that the diffusion approximation is applicable only in such large space-time scales where the dependence on z and t is essential. At smaller space-time scales, the turbulent diffusion approximation is inapplicable.

This result is based on the fact that orbits of the electrons in the (x, y) plane turn out to be closed. When the initial smooth distributions of the electron number density n and the temperature T are given, the motion along the closed orbits leads to nothing else but only averaging over the orbit. In this approximation, the values averaged over the orbits, $\bar{n}$ and $\bar{T}$, remain always different for different orbits.

Another formulation of the problem considers a layer of finite thickness, e.g., $0 < x < x_0$, where the considered two-dimensional random magnetic field is concentrated. The currents flowing along the z-direction that produce the field b are absent on both sides of this layer.

Let us prescribe the value n_1 of the number density to the left of the layer $(x < 0)$ and another value n_2 to the right $(x > x_0)$. The particle flux is given by

$$q_x = D(n_1 - n_2)/x_0 , \tag{5}$$

following the definition of coefficient of diffusion D. It is now clear that with the growth of the layer thickness x_0 the fraction of orbits which are not closed within the layer (i.e., between $x = 0$ and $x = x_0$) decreases. Therefore, the particle flux decreases more rapidly than x_0^{-1}, either as x_0^{-m} with $m > 1$ or as $\exp(-k_0 x_0)$. But this implies that there is no definite value of D and in the limit of large x_0, the effective value of D tends to zero.

A finite (non-vanishing) value of D can be obtained only when the Larmor radius r is finite.

In a similar problem of a two-dimensional steady vortex motion of incompressible fluid, the turbulent diffusivity is non-vanishing only when the molecular diffusivity k is non-vanishing (Zeldovich, 1982). In this case D is proportional to a certain fractional power of k.

In this respect, a two-dimensional steady motion is similar to a one-dimensional unsteady motion (Zeldovich, 1982). The analogous correspondence with the patterns of caustic lines (or surfaces) in problems with equal numbers of variables (e.g., either x and t or x and y) was noted by Arnold (1982).

In similar completely three-dimensional stationary problems as well as in two-dimensional non-stationary problems (three variables, either x, y, z or x, y, t), both the electron diffusivity D along the magnetic lines and the turbulent diffusivity D_t of hydrodynamic motion differ from zero even in the limits $r \to 0$ and $k \to 0$.

To verify this, consider, e.g., a non-stationary two-dimensional percolation problem for a system of magnetic lines with arbitrary unsmooth dependence of the magnetic field on time. Here a particle moves along a closed field line around some center during the time interval τ_1 and during this period its coordinates differ from their initial values by at most the orbit radius. In the following time interval τ_2, after the magnetic field has abruptly changed, the particle can jump to another center and move along a new trajectory which is independent of the previous one.

Evidently, after a few such steps, the displacement grows with the square root of time, that is, according to the typical diffusive law.

Let us consider briefly the percolation properties of a two-dimensional field consisting of two components, a steady uniform field and a randomly varying field. The weak random field bends the field lines of the uniform field only slightly. It is clear that the only physical effect is the following. In some places, where the random field is sufficiently strong, local catastrophes lead to the isolation of islands of closed streamlines associated with the local maxima of a accompanied by saddle points.

More interesting is the opposite case when a very weak uniform steady field is imposed on a given random field. It can be shown that the magnetic flux corresponding to the uniform field concentrates into narrow ropes (channels) along which occurs percolation in the direction of the uniform field, even though the ropes are tangled. Within the ropes the field strength is of the order of the r.m.s. chaotic field; weakness of the uniform field leads to the narrowness of the ropes.

Note one more particular case. Consider a uniform field in an ideally conducting medium. The random motion of the medium amplifies the field. The flux is conserved because under chaotic tangling the mean cosine of the angle between the current field direction and the original one tends to zero. The percolation remains to be ideal until the field is frozen into the medium and the field lines do not reconnect. The pattern of narrow percolation that channels here arises only after a sufficient time has elapsed from the beginning of the motion.

To conclude this discussion of percolation along the magnetic lines of a two-dimensional field, we should emphasize that although this picture seems to be quite natural, it implicitly relies on a non-trivial restriction on the magnetic field configuration. This restriction can be conveniently expressed in terms of the correlation properties of electric currents that produce the considered magnetic fields; recall that in the two-dimensional case, the current j is directed along the z-axis and is related to the potential a through

$$\Delta a = j\,.$$

It can be shown (Zeldovich, 1983) that the discussion above implicitly assumes that the electric currents are positively correlated, i.e.,

$$\langle j(\mathbf{x})\, j(\mathbf{x} + \mathbf{r})\rangle \geq 0\,. \qquad (6)$$

In particular, the discussion above is applicable when the considered magnetic field is produced by currents that are directed completely at random (are uncorrelated). However, the situation can be very different when the currents are screened and thus compensated. Percolation in a more complicated system is considered in the following section where the cosmological percolation problem serves as an example.

6.2. INTERMITTENCY AND PERCOLATION

The picture described in the previous section reflects only one aspect of percolation even though it is very important. In order to reveal another aspect, let us turn to matter percolation in the universe. Presently, the average matter density in the universe is much below the density of any cosmic body, from planets to galaxies and galaxy clusters. This implies that matter is distributed uniformly only over a large scale while at smaller scales the distribution is highly non-uniform: vast voids coexist with compact clumps of matter. The volume fraction occupied by these clumps (specifically, the galaxy clusters) is considerably smaller than the constant of percolation, 0.16. Thus, the results of Section 6.1 seem to imply that galaxy clusters should form isolated islands separated by voids. Such distribution of matter in the universe coincides with the clustering model which assumes that the observed structure of the universe is the result of progressive hierarchical clustering of matter at much larger scales. It is noticeable that a careful analysis of observations definitely rejects this model (see the review of Zeldovich and Shandarin, 1982) and it can be consistent with observations only with a very low probability. In other words, the galaxy clusters cannot be described as balls widely scattered according to the Poisson law. In order to reach a correct, rather than apparent, conclusion about matter distribution in the universe, one should remember (Zeldovich, 1983) that at the earlier stages of evolution of the expanding universe matter distribution was quite different: almost all the matter was in the state of high density, exceeding the average density of the present universe. At these epochs, the matter that now forms clusters of galaxies occupied a major fraction of the universe volume. The results of Section 6.1 then imply that percolation at that time must proceed along these dense regions while the future voids must be isolated. The present low-density state of the universe is produced from a dense state of the early universe due to expansion which can be considered here as continuous mapping. Obviously, a continuous mapping cannot affect the percolation properties of the objects, i.e., even now the percolation must occur along the system of galaxy clusters.

Thus, we have reached a paradoxial result: the concepts of percolation discussed in Section 6.1 seem to imply that both clusters and voids must be isolated in the present universe.

The paradox is solved as follows. It seems plausible that matter inhomogeneities in the early universe are approximately spherical in shape and the percolating structure of the dense regions is formed by contacts of these spherical regions. The situation is quite different at the present epoch: during the expansion of the universe, the gravitational instability, to the first approximation, compresses the matter inhomogeneities along one direction only while expansion still can occur along the remaining two directions (Zeldovich, 1970). As a result, matter is concentrated within flat, thin formations — like pancakes. Thus, the percolation theory for the present universe should incorporate an additional characteristic parameter, a small ratio of the thickness of the pancake to its diameter. The value of this parameter is considerably smaller than the value 0.16 of the three-dimensional percolation parameter; percolation properties of the matter distribution are determined by the former parameter. Gravitational instability leads to the pronounced cellular structure of the matter distribution in which large voids are separated by very thin walls that contain the principal part of the total mass.

We see that the intermittency in the distribution of a random field can qualitatively modify the percolation properties. Percolation properties of a system of thin, long strings or plates can considerably differ from those of a system with more or less spherical bodies considered in Section 6.1. This kind of percolation is very important for many applications. For example, the implantation of flat flakes of suitable impurity into a polymer can lead to a considerable slowing down of its deterioration with time. Modern advances in the theory of percolation in intermittent media are reviewed by Menshikov *et al.* (1986). The authors are much indebted to S.A. Molchanov for numerous discussions of the percolation theory reflected in this chapter.

In three-dimensional intermittent systems, percolation seems to be associated with the presence of a rather long and thin structure. For moderate ratios of the impurity bodies or for low concentrations, their role seems to be negligible. An implicit indication of this is a low value of the percolation threshold even for spherical particles, $C_m \approx 0.16$.

The situation is different in two dimensions where percolation is not connected with any small parameters. In particular, Menshikov *et al.* (1986) proposed the following picture of two-dimensional intermittent percolation: for small concentrations of the conductor phase, percolation cannot occur along it, for its large concentration stops percolation over the insulator phase. However, unlike non-intermittent distributions, for intermittent distributions transition between these two regimes occurs before the conductor phase reaches half the volume. Around the state with equal concentrations of both phases, there is a finite interval of concentrations for which percolation does not occur along either phase. The structure in these cases is similar to a set of nested alternating closed layers of conductor and insulator. What occurs is a local spontaneous violation of translational symmetry and the matter distribution resembles a polycrystal whose individual crystals consist of such nested layers. An isolated set of nested layers has a distinguished center. A nested-layer monocrystal has finite size but a large number of such monocrystals fill the pores of a larger nested-layer monocrystal. The resulting hierarchy of nested-layer monocrystals is translationally invariant (see Fig. 6.1). In finite bodies, such a structure leads to irregular changes of conductive to insulating properties under weak changes of size of samples.

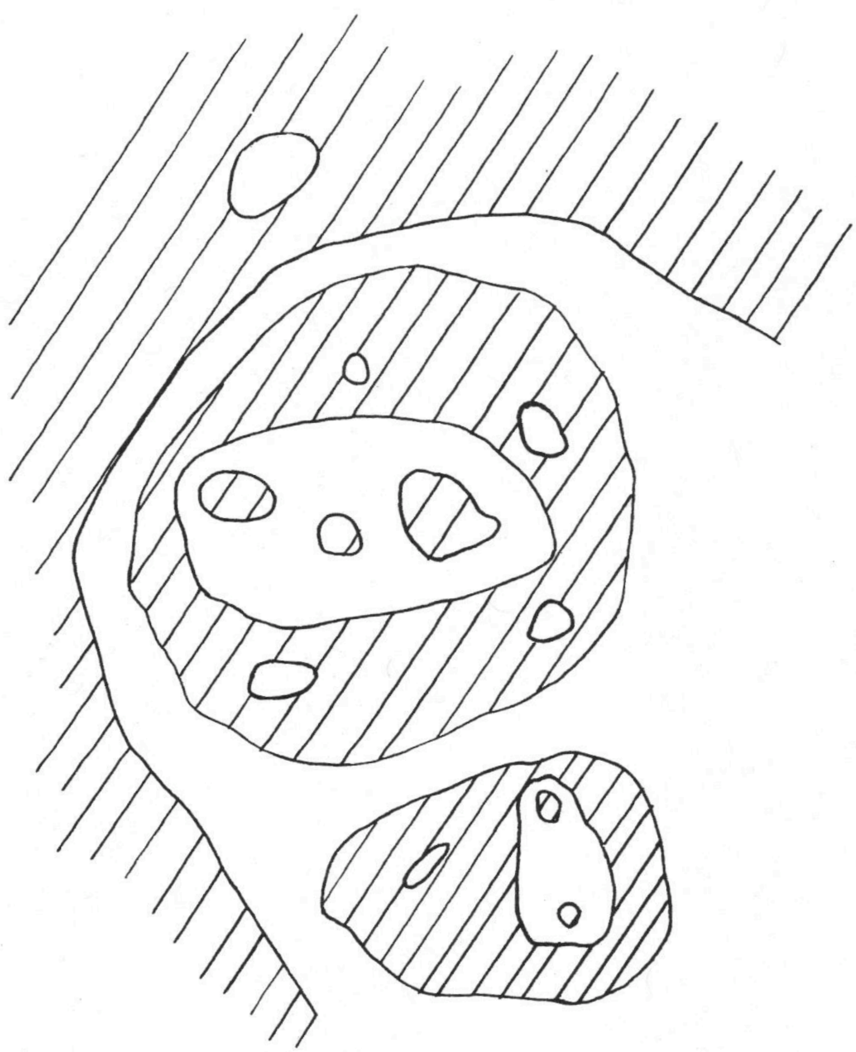

Fig. 6.1. Qualitative scheme of nested structure for which percolation is impossible over either phase in two dimensions.

These ideas can be expressed also in another way. For intermittent media, the standard ergodic concepts are typically violated. In the considered case, intermittency leads to ambiguity of the notion "percolation threshold". This threshold can be understood as a critical value of the concentration at which exists an infinite cluster of a given phase (C_H), or as the value of concentration at which the mathematical expectation value of the cluster size becomes infinite ($C_T^{(1)}$). One can also introduce an intermediate percolation threshold, $C_T^{(p)}$ for which the p-th moment of the cluster size becomes infinite. Finally, one more quantitative characteristic of the percolation can be introduced, that is the value of the concentration C_s for which

$$\lim_{l \to \infty} P(l) = 0 ,$$

where $P(l)$ is the probability of percolation through a region of size l. Clearly,

$$C_s \leqslant \ldots \leqslant C_T^{(p+1)} \leqslant C_T^{(p)} \ldots \leqslant C_T^{(1)} \leqslant C_H . \tag{7}$$

These relations among the differently defined percolation thresholds can be compared with the relations among the growth rates of the magnetic field moments in a typical realization in a random medium (see Chapter 9).

Differences in the various thresholds can be considered as a criterion of the intermittent character of percolation. Modern theory has not as yet studied relations between differently defined thresholds in full detail. More or less understood is not the problem of percolation on a continuous plane but the more artificial problem of percolation on a metal lattice with randomly scattered inclusions of insulator. Kesten (1982) has proved that if these inclusions are uncorrelated, then all kinds of thresholds exactly coincide. Menshikov *et al.* (1986) conjectured that for Poisson random fields, the percolation thresholds differ due to well pronounced correlations. For Gaussian fields, intermittent percolation, i.e., strict inequalities in (7), seems to be impossible, at least when the field correlation function is non-negative.

In its initial formulation, the percolation theory considers percolation along purely random regions. This theory is related to another realm of problems in which the presence of some degree of regularity is essential, e.g., in the case of regions with regular but randomly deformed boundaries. Such regions are exemplified by deformed or imperfect resonators. The theory of such resonators has some common aspects with the theory of percolation, and we shall briefly consider them in the following section (Zeldovich and Sokoloff, 1983).

6.3. EIGEN-OSCILLATIONS OF A REGION WITH RANDOM BOUNDARY

Modern mathematical physics considers the analysis of generic problems (see, e.g., Arnold, 1983) very important. These problems often drastically differ from exactly solvable problems where solvability is associated with a high degree of symmetry. One particular case of a generic problem is the problem with statistically (randomly) determined parameters.

Here we consider the question of eigen-oscillations of a region with randomly determined boundary, i.e., the question of eigenfunctions of the problem

$$\Delta u = -\lambda^2 u, \quad u_{|\partial \mathscr{G}(\omega)} = 0\,,$$

where $\partial\mathscr{G}(\omega)$ is the random boundary and ω is the random parameter (Babich and Buldyrev, 1972).

First consider a one-dimensional problem where the random region is a line segment $[a(\omega), b(\omega)]$. Although the solutions to this problem are random functions, the n-th eigenfunction has, like the deterministic one, $n - 1$ zeros in the interval (a, b) so that the distance between the zeros is of the order of $(b - a)/n$. In the one-dimensional case, the requirement of generic properties does not introduce novel properties into the structure of the zeros of an eigenfunction. The situation is more complicated in multi-dimensional cases.

Consider now a two-dimensional region $\mathscr{G}(\omega)$ whose boundary is close to a regular curve of diameter $2R$, e.g., to a circle of radius R or a square of side R. This regular curve is disturbed by a random function which is locally similar to the Wiener process, i.e., $\Delta\xi \sim (\Delta l)^{1/2}$, where $\Delta\xi$ is the deviation from the unperturbed curve and Δl is the coordinate measured along the regular curve.

The eigenfunctions have zero lines within the region $\mathcal{G}(\omega)$, which are analogous to the zeros of one-dimensional eigenfunctions. The zero line patterns can be used for topological classification of the eigenfunctions. Excitation of the next higher oscillation mode adds an extra zero line. Due to the random nature of the boundary, this new line with probability unity does not intersect other zero lines. Indeed, level lines $u = 0$ of a random function $u(r)$ have no singular points. The intersection of two zero-level contour lines is not a generic event in the random media, i.e., in the space of the surfaces the measure corresponding to this event is zero. A formal proof of this fact can be found in Brüning (1978). Therefore, n zero lines divide the region $\mathcal{G}(\omega)$ into $n + 1$ parts. These zero lines must either be close to or intersect the region boundary. Of course, in contrast to the one-dimensional case, the number of zero lines now cannot completely describe an eigenfunction; an accurate classification should be taken into account as well as the topology of the network of zero lines.

We should stress that the absence of any symmetry in the shape of the considered region implies that the distance between zero lines is of the order of R/n, i.e., zero lines can approach each other closely only with a small probability or for large n.

When there is some symmetry, the set of zero lines can have a more complicated structure. Due to symmetry, zero lines can have numerous mutual intersections so that $m + n$ zero lines parallel to the sides of a regular square divide the square into $(m + 1)$ $(n + 1)$ regions (see Fig. 6.2). R. Courant (see Hilbert and Courant, 1981) has obtained an estimate of the number of such regions as a function of the eigenfunction's order. He has also noted that in degenerate situations, including the problem of oscillations in a square, only a few very symmetric eigenfunctions provide that large number, $(m + 1)$ $(n + 1)$, of regions separated by zero lines. A linear combination of eigenfunctions with generic coefficients behaves similarly to an eigenfunction of a statistical problem. The intersection points of zero lines are destroyed while the zero lines become closed. However, in this case the intersections of zero

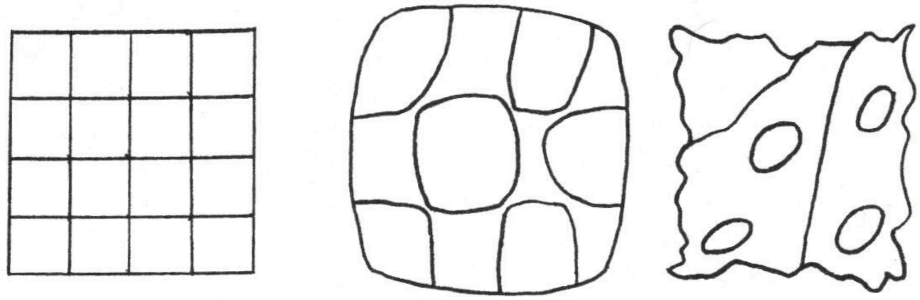

Fig. 6.2. Schematic structure of zero lines of eigenfunctions in problems (a) with symmetry; (b) with weak deviation of symmetry due to boundary perturbation; (c) with strongly violated symmetry.

lines are destroyed due to violation of the symmetry by the coefficients of the linear combination of symmetric eigenfunctions, rather than due to violation of the region's symmetry.

There exists a class of perturbations of a symmetric region that leads to, in a certain sense, weak perturbation of the pattern of zero lines even though the intersection points are destroyed (see, e.g., Babich and Buldyrev, 1972; Lazutkin, 1979). It turns out that when a circle is perturbed in such a way that the curve remains convex, the distance between neighbouring closed zero lines is exponentially small in the perturbation, rather than of the order of R/n. In this case, perturbation of the boundary is too weak to lead to a considerable change of the zero lines pattern.

One can also analyze the opposite case of very strong perturbations which drastically change the picture. This change is associated with the fact that, apart from eigen-oscillations localized far from the boundary, which were considered above, there arise in a region with random boundary special boundary-layer eigen-oscillations of the type of Echo Wall.[b] Their existence is associated with the irregularities of the boundary and interference of the eigenwaves at these irregularities. Such

[b] The Echo Wall, known as the Whispering Galleries in Russian, a masterpiece of Chinese architecture in Beijing, more than adequately describes the phenomenon. A whispering sound propagates very far along these walls, to distances where one cannot normally hear a loud voice originating from the same location (Fig. 6.3).

Fig. 6.3. The Echo Wall in Beijing always attract tourists (the illustration from Peng Zhen, 1986).

oscillations decay exponentially with distance from the boundary and the question of the structure of zero lines is of minor importance for them. However, if the surface diameter $\sim 2R$ is much greater than the perturbation amplitude and, in addition, the perturbed boundary length does not differ by an order of magnitude from $2R$, then "whispering" perturbations are negligibly deep within the considered region. The situation is different in three (and more) dimensions. In such a case, the boundary area ($\sim R^2$) grows faster than its diameter and the large area of the boundary leads to the plausible appearance of high peaks of height $\sim R$ even if the probability of such a peak is low. Therefore, in the three-dimensional case, the dependence required for the damping of boundary-layer perturbations is $\Delta\xi \sim (\Delta l)^{3/2}$.

Let us now discuss how these problems are connected with the percolation theory. To do this, one should consider a double limit case when both the region size and order of the eigenfunction simultaneously tend to infinity so that the characteristic value of the wave number constant. In this case, the problem reduces to that of a random function φ with zero mean value whose spectrum is concentrated in the "sphere" $|K| = K_0$ (see Section 6.1).

Now we are able to formulate the percolation problem, i.e., the question whether or not it is possible to pass from infinity to infinity, e.g., along the regions where the eigenfunction φ is positive. In three dimensions, the possibility of percolation is typical. For a very large ε, exceeding some critical value ε_0, percolation is impossible along the region with $\varphi > \varepsilon$; for $\varepsilon_0 > \varepsilon > -\varepsilon_0$ percolation occurs over both regions with $\varphi > \varepsilon$ and $\varphi < \varepsilon$; for smaller ε percolation along the region with $\varphi < \varepsilon$ is impossible. Roughly, the value of ε_0 is of the order of $\langle\varphi^2\rangle^{1/2}$ which means that for a Gaussian random variable the region with $\varphi > \varepsilon_0$ occupies 1/6 of the total volume. This problem is much more complicated in two dimensions where simultaneous percolation along both phases is impossible. In the two-dimensional case, the simplest possibility is the following. For any $\varepsilon > 0$ the lines $\varphi = \varepsilon$ and $\varphi = -\varepsilon$ delineate a system of islands with positive and negative values of φ. There can exist a system of lakes with φ of opposite sign within any island. When ε tends to zero, the picture remains qualitatively the same: the sea between the islands reduces to a system of channels but the islands remain isolated. However, one can envisage another intermittent picture: at a certain value of ε a "phase transition" occurs and the system of islands and channels is replaced by a nested-layer structure with islands of smaller size nested in larger islands with opposite sign of φ so that the whole island hierarchy is infinite.

6.4. THE ZEROS OF EIGENFUNCTIONS OF FREE AND FORCED OSCILLATIONS

The question of the structure of the manifold in which the oscillation amplitude in a resonator vanishes is also close to the percolation problem. In the simplest case, for a three-dimensional resonator this manifold consists of surfaces. However, in the generic case the oscillation amplitude vanishes at one-dimensional manifold lines which are closed or otherwise begin and end at the region boundary. After traversing along a closed zero-amplitude line, the oscillation phase changes by $\pm 2\pi$.

The appearance of topological singularities of filamentary structure whose traversal changes the phase by $\pm 2\pi$ is typical of many physical problems. Polyakov (1975) has noticed such singularities in the theory of spontaneous symmetry breaking. Consider a complex scalar field ψ. For the fields that appear in the cosmological inflation theory, the vacuum corresponds to $\psi = \psi_0 \neq 0$. If the phase changes by $\pm 2\pi$ after traversing along a closed contour, this contour must enclose a line (string) at which $\psi = 0$. Similar ideas can be developed for auto-oscillatory systems. Aldushin *et al.* (1980) consider the surface of a flame in the case where its propagation at constant speed is unstable, while nonlinear effects stabilize oscillations of the flame front at a certain amplitude and phase and the transverse dissipation equalizes the phase over the whole flame surface. If the initial conditions are such that at a certain contour on the flame surface the phase changes by $\pm 2\pi$, then this contour must enclose a point with a vanishing oscillation amplitude. This singularity occurs at a point rather than a line because the flame surface is two-dimensional. Singular lines are typical of three-dimensional auto-oscillatory systems. Similar problems appear in the theory of chemical auto-oscillatory systems and in the theory of nearly parallel rays (Berry and Nye, 1974; Ivanitsky *et al.*, 1978; Baranova and B. Zeldovich, 1981).

Similar lines also arise in the simple case of a linear oscillatory system with periodic external forcing or dissipation.

To verify this, recall that the equation of oscillations supplemented by the boundary condition $u = 0$ at a closed surface represents a self-adjoint problem for the equation $\Delta u = -\omega^2 u$. Therefore, its solutions are real. Even if some eigenfunctions u_n are complex, the conjugated functions $\bar{u}_n$ are also solutions corresponding to the same eigenvalue ω; these pairs of eigenfunctions can be represented as pairs of real functions whose zeros lie on a surface. A similar result applies to the case of phased forcing, i.e., when the condition $u(s, t) = R(s) \exp(i\omega t)$ with real R applies to some part of the boundary surface.

Let us now turn to the case where zero surfaces are replaced by zero lines. The case of forced oscillations with unphased excitation,

$$R(s) = R_1(s) + iR_2(s), \quad \frac{R_1}{R_2} \neq \text{const.} \tag{8}$$

is simple and clear. Indeed, due to the linearity of the problem solutions have the form

$$u(t, \mathbf{r}) = \exp(i\omega t)\,[\rho_1(\mathbf{r}) + i\rho_2(\mathbf{r})]\,,$$

where ρ_1 and ρ_2 correspond to R_1 and R_2, respectively. Physical solution is given by the real part of this expression, $\mathrm{Re}\,u = \rho_1 \cos \omega t + \rho_2 \sin \omega t$, and the condition $u = 0$ leads now to two conditions, $\rho_1 = 0$ and $\rho_2 = 0$. Thus, the zero manifolds are now intersections of two surfaces, i.e., the lines. Note that for $R_1 \propto R_2$ we have $\rho_1 \propto \rho_2$ and $\mathrm{Re}\,u \propto \cos(\omega t + \varphi)$ with φ constant. Thus, in this case the zero manifold is a surface, and it reduces to a line only for $R_1/R_2 \neq$ const.

The situation is the same for oscillatory systems with damping or radiation. The oscillations are now described by the equation $u_{tt} = \hat{L}u$ where $\hat{L}$ is the differential operator. Consider a boundary-value problem for this equation which is not self-adjoint, e.g., due to specific boundary conditions. Evidently, in a generic case the eigenfunctions are complex-valued. Therefore, the amplitude of oscillations turns to zero on a line or, in the case of an arbitrary number of dimensions, on a manifold with co-dimension 2 (co-dimension is the difference between dimensions of the space and of the embedded surface).

V.I. Arnold has drawn our attention to the fact that the problem of zero manifolds of eigen-oscillations was discussed by mathematicians, in particular, problems like the oscillations of two-dimensional surfaces in four-dimensional space. However, this direction has not been developed too far. The reason for this is probably associated with the fact that in classical problems of mathematical physics, even in the presence of non-self-adjoint operators, the situation usually is not generic and complex-valued frequencies correspond to real-valued eigenfunctions.

Let us illustrate the situation with examples. Consider dissipation described by the term au_t. Then the boundary value problem for the oscillations is formulated as

$$(-\omega^2 + ia\omega)u = \Delta u, \quad \frac{u}{\Gamma}\Big|_b = 0$$

and, for constant a, the substitution $\lambda^2 = \omega^2 - ia\omega$ reduces the problem to one of undamped oscillations. Based on the linearized Navier-Stokes equation, let us now describe dissipation through the term $\nu(\Delta u)_t$. Then the corresponding boundary-value problem is given by

$$-\omega^2 u = \Delta u + i\omega\nu\,\Delta u; \quad \frac{u}{\Gamma}\Big|_b = 0\,.$$

For constant ν the substitution $\lambda^2 = \omega^2(1 + i\omega\nu)^{-1}$ again reduces the problem to a dissipationless one. Finally, consider a parabolic equation $u_t = (\nu + i\Omega)\Delta u$. For $u \propto \exp(\gamma t)$, the substitution $\lambda = \gamma/(\nu + i\Omega)$ again makes the problem self-adjoint in a homogeneous and isotropic case. In these cases the complex-valued solutions arise only when, e.g., the dissipation is inhomogeneous [cf. condition $R_1/R_2 \neq$ const. in (8)]. The zero lines also arise when the excited waves leave the considered region through a hole.

Consider now another case of waves in an infinite medium:

$$u(t, \mathbf{r}) = r^{-1}\exp(i\omega t - ikr)\,.$$

Due to planar symmetry, this complex solution differs from zero everywhere. However, for the superposition of a spherically symmetric spreading wave and a travelling planar wave (which introduces a distinguished direction at infinity):

$$u(t, \mathbf{r}) = [Ar^{-1}\exp(-ikr) + B\exp(ikx)]\exp(i\omega t)\,,$$

we see that the amplitude is zero when the two conditions are fulfilled: first, $r = r_0 = A/B$ and second, on the sphere $|\mathbf{r}| = r_0$ the phases $r - x = 2\pi n/k$ with $n = 0, 1, 2, \ldots$ coincide up to 2π. The zero lines are of a few (with the number dependent on the value of kr_0) parallels on the spheres determined by the wave front. This group of problems is important in the studies of weakly divergent light beams (Baranova and Zeldovich, 1986).

All these results rely heavily on the monochromatic nature of oscillations. Qualitatively new physical effects can be associated with quasimonochromatic oscillations. An important example is provided by the Langmuir plasma oscillations whose wavelength is much greater than the Debye wavelength. As a result, there arises a natural distinction between fast and slow oscillations and the physical picture may reduce to a slow drift of the zero lines of fast oscillations.

6.5. THE MATHEMATICAL LANGUAGE OF THE PERCOLATION THEORY

Traditionally, mathematics is considered as a science of quantitative relations and spatial patterns. These two realms of mathematics were not always in a harmonious combination and relation. In the second half of the 19th century, it seemed that The Quantity had won this competition. Treatises on geometry had got rid of drawings and figures, geometrical problems had been translated into the language of equations and methods of mathematical physics had merged with the theory of differential equations. The Form won back in early 20th century when topology appeared. In this non-quantitative science, even an invariant is not necessarily a number. The modern development of mathematics can be compared to a symphony with two dominating colliding themes. One theme is the quantitative mathematics which becomes more and more computerized, pays less attention to the rigorousness of proofs and approaches further the applied sciences and engineering. Another theme is the non-quantitative mathematics — geometry, topology, etc. This branch of mathematics has found many unexpected common points with the humanities and arts. It is still developing adequate language of self-expression and is, therefore, much concerned with the rigorousness of its proofs. However, this branch also makes its first steps toward computers, declaring that the numerical nature of computer science represents only the embryonic state of development of this science. Applications of non-quantitative mathematics to physics and other natural sciences are still not numerous even though their number grows steadily. In this respect, the mathematical method of percolation theory is unique, being almost completely based on the achievements of non-quantitative mathematics. Unfortunately, it is a tradition of this field of thought to express the results in a very abstract way paying much attention to rigorous proofs. As a result, books on percolation theory written by mathematicians often

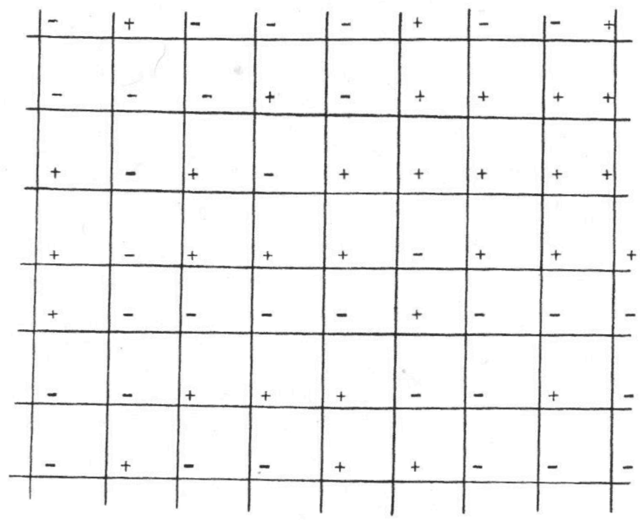

Fig. 6.4. In the knots problem, the knots conduct electric current with probability p (plus signs) and are insulating with probability $q = 1 - p$ (minus signs).

can only be understood by their authors. Percolation mathematicians are more than suspicious about the results obtained with the help of computers and persistently argue that simple and clear results obtained in percolation theory by physicists may prove to be inaccurate, having neglected a whole world of complex percolation phenomena like, e.g., intermittency. Hopefully, a synthesis of physical and mathematical approaches to percolation theory will be reached in the foreseeable future. But now we can only reflect this complicated state of lack of mutual understanding and give only a hint on specific methods of percolation theory.

The first step that mathematicians take in the studies of percolation is the discretization of space. Instead of percolation along a region in which a random field exceeds a given value, one considers percolation along a regular lattice with certain links removed with a given probability. It turns out that the result strongly depends on the manner of discretization, i.e., on whether one considers a regular square, triangular, hexagonal or more complicated lattices. The links also can be removed in different ways. For example, one can leave a chain crossing unaffected with probability p and remove it with probability $q = 1 - p$. This discretized problem is called the knot problem (see Fig. 6.4). Otherwise, one can leave a link unaffected with probability p and remove it with probability $q = 1 - p$ (Fig. 6.5). This is the link problem. These two

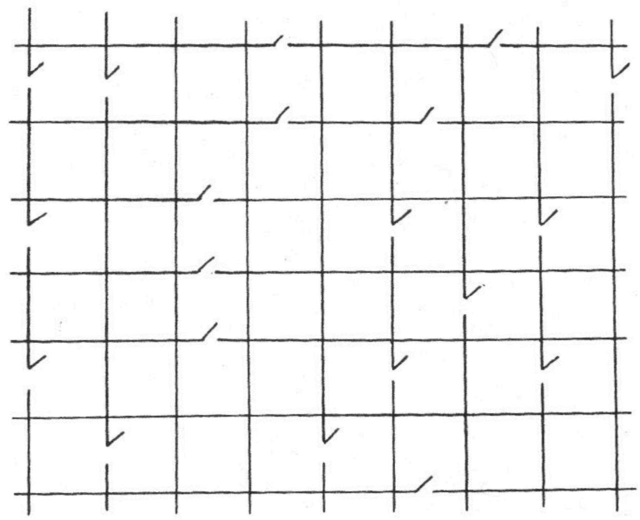

Fig. 6.5. In the links problem, the square lattice (solid lines) and the dual one (dashed lines) are shifted by a half-period with respect to each other along both axes.

kinds of formulation are seemingly analogous and one would expect that percolation thresholds (i.e., those critical values p_{cr} that correspond to marginal percolation) are close to each other. However, the actual link problem possesses higher symmetry than the knot problem. It turns out that the destructive role of a removed crossing is more pronounced than that of a removed link. To verify this, one should formulate the link problem in a symmetric form. This is achieved by the construction of the so-called dual lattice (Fig. 6.6) which is composed of connections between the cell centers of the initial lattice. A crossing of the dual lattice is considered present (switched on) when it crosses a removed crossing of the initial lattice and vice versa. It can be seen easily that if percolation occurs from the left to the right along the switched-on links of the initial lattice, then removed links of the dual lattice are percolating from above to below and vice versa. The absence of percolation along the switched-on links of the initial lattice implies percolation along the removed links of the dual one. Unfortunately, it is very difficult to turn this intuitive knowledge into a rigorously proved result (see Kesten, 1986). The existence of such symmetry implies that percolation along the switched-on links is immediately followed by percolation along the removed links when the critical value of probability is passed (i.e., the intermittent percolation is impossible). Since the properties of the initial and dual lattices are symmetric, this means that

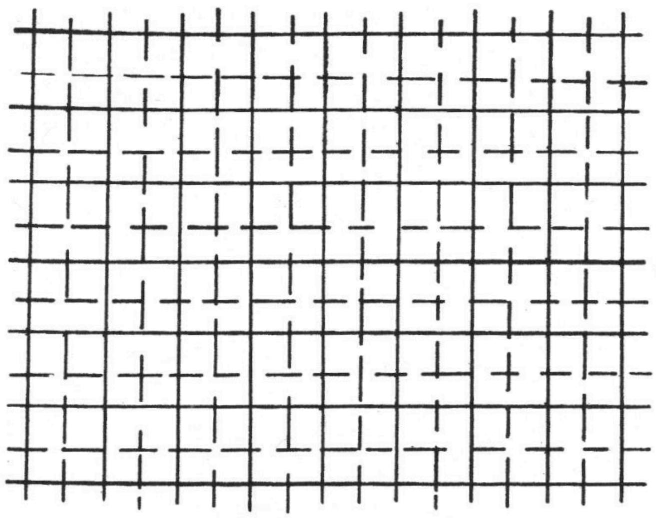

Fig. 6.6. A dual lattice composed of connections between the cell centers of quadratic lattice.

$$p_{cr} = 0.5 .$$

There is no such symmetry for the knot problem and a similar procedure brings us to a dual lattice whose percolation properties differ from those of the initial one. For this reason the percolation threshold exceeds one half in the knot problem. Numerical experiments give the value

$$p_{cr} = 0.59 .$$

Those cases where percolation thresholds can be estimated at the mathematical level of rigor, without the aid of computers, are also based on symmetry arguments, even though they are much more complicated and subtle than in the problem considered above. Sometimes the problem can be reduced to some equations. For instance, symmetry arguments lead to the conclusion (see Efros, 1982, pp. 171–172) that the percolation thresholds for the link problem on a triangular lattice obeys the equation

$$p_{cr}^{\,3} - 3p_{cr} + 1 = 0 .$$

Verification of the fact that this equation has only one root

$$p_{cr} = 2 \sin \frac{\pi}{18}$$

belonging to the interval $0 < P_{cr} < 1$ or an approximate estimation of this root is a problem of quantitative mathematics, and the estimation can be obtained easily with the help of a computer. However, these computational problems are incomparably easier than those that arise in an attempt at direct numerical estimation of the percolation threshold.